SELF-SIMILAR PROBLEMS OF THREE-DIMENSIONAL BOUNDARY LAYER IN THE PRESENCE OF SUCTION OR BLOWING

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Velocity distribution functions are obtained for the external boundary of a boundary layer and also for the flow of a liquid along a permeable surface; in these cases the integration of the initial partial differential equations that describe the motion of the liquid in a three-dimensional boundary layer in a laminar regime can be reduced to the integration of a system of ordinary differential equations. Results are presented for the numerical solution of one of the cases of a self-similar three-dimensional laminar boundary layer, performed on a Minsk-22 computer.

1. The problem of determining the characteristics of the motion of an incompressible liquid in a three-dimensional laminar boundary layer on a cylindrical permeable surface reduces to the integration of the system of partial differential equations

$$u_1\frac{\partial u_1}{\partial x_1}+u_2\frac{\partial u_2}{\partial x_2}+u_3\frac{\partial u_3}{\partial x_3}=U_1\frac{\partial U_1}{\partial x_1}+U_2\frac{\partial U_1}{\partial x_2}+\nu\frac{\partial^2 u_1}{\partial x_3^2} \quad (1.1)$$

$$u_1\frac{\partial u_2}{\partial x_1}+u_2\frac{\partial u_2}{\partial x_2}+u_3\frac{\partial u_2}{\partial x_3}=U_1\frac{\partial U_2}{\partial x_1}+U_2\frac{\partial U_2}{\partial x_2}+\nu\frac{\partial^2 u_2}{\partial x_3^2} \quad (1.2)$$

$$\partial u_1/\partial x_1+\partial u_2/\partial x_2+\partial u_3/\partial x_3=0 \quad (1.3)$$

for the following boundary conditions:

$$\begin{gathered} u_1=0,\quad u_2=0,\quad u_3=v_0 \quad \text{for} \quad x_3=0 \\ u_1=U_1(x_1,x_2),\quad u_2=U_2(x_1,x_2) \quad \text{for} \quad x_3=\infty \end{gathered} \quad (1.4)$$

where the x_1 and x_2 axes of a Cartesian coordinate system are positioned on the surface of the body, the x_3 axis is perpendicular to it, v_0 is the flow velocity of the liquid through the permeable surface ($v_0 > 0$ for blowing, $v_0 < 0$ for suction).

To find the self-similar solutions of the system of equations (1.1)-(1.3) we convert to dimensionless quantities, introducing characteristic scales of the length L and of the velocity U_0, determining the Reynolds number $\mathrm{Re}=U_0L/\nu$ of the flow.

We take the transverse coordinate in the form

$$\eta=x_3\sqrt{\mathrm{Re}}/(Lf_1(x_1)f_2(x_2)) \quad (1.5)$$

where $f_1(x_1)$ and $f_2(x_2)$ are dimensionless scale factors, which allow us to perform a similarity transformation for all the velocity profiles in the boundary layer.

These factors are to be determined along with the distributions of the components $U_1(x_1, x_2)$ and $U_2(x_1, x_2)$, the velocities of the potential flow, which by analogy, will be investigated in the form

$$U_1(x_1,x_2)=V_1(x_1)V_2(x_2),\quad U_2(x_1,x_2)=W_1(x_1)W_2(x_2) \quad (1.6)$$

2. Using the boundary conditions (1.4), from Eq. (1.3) we obtain for the velocity component $u_3(x_1, x_2, x_3)$

$$u_3=v_0-\int_0^{x_3}\left(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}\right)dx_3 \quad (2.1)$$

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TABLE 1

№	V_1/α	V_2/γ	W_1	W_2	f_1/λ	f_2/ζ	$\frac{-v_0}{GU_0}\sqrt{\mathrm{Re}}$
1	x_1^m	1	$\alpha(1-m)x_1^{m-1}$	γx_2	$x_1^{\frac{1-m}{2}}$	1	$x_1^{\frac{m-1}{2}}$
2	x_1	x_2^{n-1}	α	$\frac{\gamma}{1-n}x_2^n$	1	$x_2^{\frac{1-n}{2}}$	$x_1^{\frac{n-1}{2}}$
3	x_1^m	x_2^{-m}	αx_1^{m-1}	γx_2^{1-m}	$x_1^{\frac{1-m}{2}}$	$x_2^{\frac{-m}{2}}$	$x_1^{\frac{m-1}{2}}x_2^{\frac{m}{2}}$
4	x_1	1	β	εx_2	1	1	1
5	x_1^m	1	βx_1^{1-m}	ε	$x_1^{\frac{1-m}{2}}$	1	$x_1^{\frac{m-1}{2}}$
6	1	x_2^n	β	εx_2^{1-n}	1	$x_2^{\frac{n}{2}}$	$x_2^{\frac{-n}{2}}$
7	x_1^m	1	β	ε	$x_1^{\frac{1-m}{2}}$	1	$x_1^{\frac{m-1}{2}}$
8	1	1	β	εx_2^n	1	$x_2^{\frac{1-n}{2}}$	$x_2^{\frac{n-1}{2}}$
9	e^{mx_1}	1	β	ε	$e^{\frac{-mx_1}{2}}$	1	$e^{\frac{mx_1}{2}}$
10	1	1	β	εe^{nx_2}	1	$e^{\frac{-nx_2}{2}}$	$e^{\frac{nx_2}{2}}$
11	e^{mx_1}	1	βe^{-mx_1}	ε	$e^{\frac{-mx_1}{2}}$	1	$e^{\frac{mx_1}{2}}$
12	1	e^{nx_2}	β	εe^{-nx_2}	1	$e^{\frac{nx_2}{2}}$	$e^{\frac{-nx_2}{2}}$
13	e^{mx_1}	1	$-\alpha m e^{mx_1}$	γx_2	$e^{\frac{-mx_1}{2}}$	1	$e^{\frac{mx_1}{2}}$
14	x_1	e^{nx_2}	α	$-\frac{\gamma}{n}e^{nx_2}$	1	$e^{\frac{-nx_2}{2}}$	$e^{\frac{nx_2}{2}}$

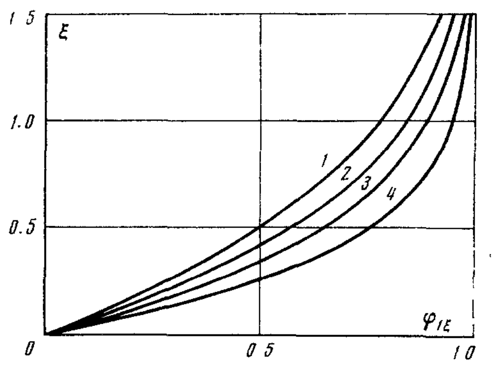

Fig. 1

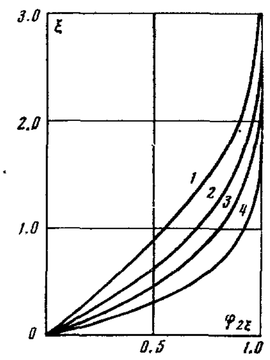

Fig. 2

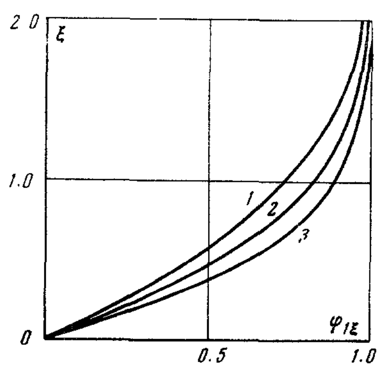

Fig. 3

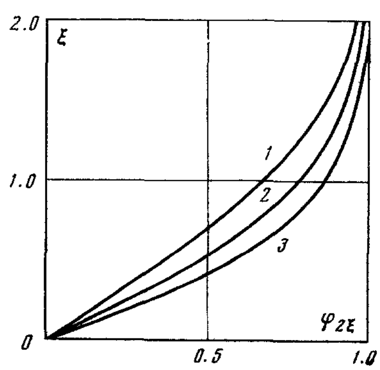

Fig. 4

When self-similar motions exist, the velocity components $u_1(x_1, x_2, x_3)$ and $u_2(x_1, x_2, x_3)$ are determined by

$$u_1 = V_1(x_1) V_2(x_2) F_{1\eta}(\eta), \quad u_2 = W_1(x_1) W_2(x_2) F_{2\eta}(\eta) \tag{2.2}$$

where $F_1(\eta)$ and $F_2(\eta)$ are unknown functions of the dimensionless transverse coordinate η.

After taking into account (1.5) and (2.2), we can reduce Eq. (2.1) to the form

$$u_3 = v_0 + \frac{U_0}{\sqrt{\mathrm{Re}}\, f_1 f_2} \{c_1 [\eta F_{1\eta}(\eta) - F_1(\eta)] - a_1 F_1(\eta) + g_1 [\eta F_{2\eta}(\eta) - F_2(\eta)] - k_1 F_2(\eta)\} \tag{2.3}$$

Substituting relations (2.2) and (2.3) into (1.1) and (1.2), we arrive at a system of equations for determining the functions $F_1(\eta)$ and $F_2(\eta)$.

$$\begin{aligned} &F_{1\eta\eta\eta} + a_1 (1 - F_{1\eta}^2 + F_1 F_{1\eta\eta}) + c_1 F_1 F_{1\eta\eta} + b_1 (1 - F_{2\eta} F_{1\eta}) + (g_1 + k_1) F_2 F_{1\eta\eta} + G F_{1\eta\eta} = 0 \\ &F_{2\eta\eta\eta} + k_1 (1 - F_{2\eta}^2 + F_2 F_{2\eta\eta}) + g_1 F_2 F_{2\eta\eta} + b_2 (1 - F_{1\eta} F_{2\eta}) + (c_1 + a_1) F_1 F_{2\eta\eta} + G F_{2\eta\eta} = 0 \end{aligned} \tag{2.4}$$

$$\begin{aligned} &a_1 = \frac{L}{U_0} f_1^2 f_2^2 V_1' V_2, \quad c_1 = \frac{L}{U_0} f_1 f_1' f_2^2 V_1 V_2, \quad b_1 = \frac{L}{U_0} f_1^2 f_2^2 \frac{V_2'}{V_2} W_1 W_2 \\ &g_1 = \frac{L}{U_0} f_1^2 f_2 f_2' W_1 W_2, \quad k_1 = \frac{L}{U_0} f_1^2 f_2^2 W_1 W_2', \quad b_2 = \frac{L}{U_0} f_1^2 f_2^2 V_1 V_2 \frac{W_1'}{W_1} \\ &G = -\frac{v_0}{U_0} \sqrt{\mathrm{Re}}\, f_1 f_2 \end{aligned} \tag{2.5}$$

where the prime denotes differentiation of the function with respect to the variable on which this function depends.

The boundary conditions of the system of equations obtained are written in agreement with (1.4), (2.2), and (2.3).

$$\begin{aligned} &F_1 = 0,\ F_2 = 0,\ F_{1\eta} = 0 \quad \text{for } \eta = 0 \\ &F_{1\eta} = 1,\ F_{2\eta} = 1 \quad \text{for } \eta = \infty \end{aligned} \tag{2.6}$$

3. Solving the system of differential equations (3.4) for the unknowns f_i, V_i, and W_i, we can determine for what laws of variation of velocity of the external potential flow do the self-similar motions of the liquid in the boundary layer hold.

The distributions obtained for the velocities V_i and W_i should, as was noted in [1], satisfy the equations of motion of a nonviscous liquid, which for the given problem gives the condition

$$2V_1V_1'V_2V_2' + V_1V_2'W_1W_2' - V_1'V_2W_1'W_2 + V_1W_2\,(V_2''W_1 - V_2W_1'') - 2W_1W_1'W_2W_2' = 0 \tag{3.1}$$

All the velocity distribution laws for an external flow and the distribution laws for the velocity of suction (blowing) satisfying the conditions (3.1) and (2.5) are represented in Table 1.

4. As an example we consider the particular case of a three-dimensional boundary layer on a permeable surface, for which the following self-similar solution holds:

$$\begin{aligned} &V_1 = \alpha x_1^m, \quad V_2 = \gamma, \quad W_1 = \alpha (1-m) x_1^{m-1}, \quad W_2 = \gamma x_2 \\ &f_1 = \lambda x_1^{(1-m)/2}, \quad f_2 = \zeta, \quad \frac{v_0}{U_0} \sqrt{\mathrm{Re}} = -G x_1^{(m-1)/2} \end{aligned} \tag{4.1}$$

where α, γ, λ, ζ, and G are constants.

The case of an impermeable surface ($v_0 = 0$) was considered in [1].

After determining the coefficients of Eqs. (2.4) based on Eqs. (2.5), after making the substitution of variables

$$F_1(\eta) = \sqrt{\frac{2m}{a_1(m+1)}}\,\varphi_1(\xi), \qquad F_2(\eta) = \sqrt{\frac{2m}{a_1(m+1)}}\,\varphi_2(\xi)$$

$$\eta = \sqrt{\frac{2m}{a_1(m+1)}}\,\xi$$

we obtain the system of equations

$$\varphi_{1\xi\xi\xi} + \varphi_1\varphi_{2\xi\xi} + \frac{2(1-m)}{1+m}\varphi_2\varphi_{1\xi\xi} + \frac{2m}{m+1}(1 - \varphi_{1\xi}^2) + Q\varphi_{2\xi\xi} = 0 \tag{4.2}$$

$$\varphi_{2\xi\xi\xi} + \varphi_1\varphi_{2\xi\xi} - \frac{2(1-m)}{1+m}(\varphi_{2\xi}^2 - \varphi_2\varphi_{2\xi\xi} - \varphi_{1\xi}\varphi_{2\xi}) + Q\varphi_{2\xi\xi} = 0.$$

$$Q = G\sqrt{\frac{2m}{a_1(m+1)}}$$

The system (4.2) was numerically integrated on a Minsk-22 computer for the boundary conditions

$$\begin{aligned} &\varphi_1 = 0,\ \varphi_2 = 0,\ \varphi_{1\xi} = 0,\ \varphi_{2\xi} = 0 \quad &\text{for } \xi = 0 \\ &\varphi_{1\xi} = 1,\ \varphi_{2\xi} = 1 \quad &\text{for } \xi = \infty \end{aligned} \tag{4.3}$$

The results of calculating the velocity distribution in the boundary layer for several parameters of the problem are shown in Figs. 1-4. The value of the parameter n was 1 for the curves of Figs. 1 and 2, and was $^1/_3$ for the curves of Figs. 3 and 4. The curves 1, 2, 3, and 4 correspond to Q = 0, 0.5, 1.0, and 2.

LITERATURE CITED

1. Yu. E. Karyakin, "Self-similar problems of a three-dimensional boundary layer," Tr. Leningr. Politekhn. In-ta, No. 243 (1965).